On the Properties of a Cauchy Problem Solution of a Sobolev Type System

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Abstract

The uniqueness of the solution of a Cauchy problem of a Sobolev type system of partial differential equations is established and the Lp-estimates of the solution are obtained.

Key words: Duhamel’s principle, Heat conduction, Lp estimates, Marcinkiewicz theorem, Uniqueness

Introduction

The eminent Russian mathematician S.L. Sobolev had studied the following system of partial differential equations (Sobolev 1954):

\[
\begin{aligned}
\frac{\partial \bar{v}}{\partial t} - \left[ \bar{v}, \bar{\omega} \right] + \nabla P = \bar{f}(x, t) \\
\text{div } \bar{v} = 0
\end{aligned}
\]  

subjected to the initial conditions

\[
\begin{aligned}
\bar{v}(x, 0) = \bar{v}^0(x) \\
\text{div } \bar{v}^0(x) = 0
\end{aligned}
\]  

The above system (1) is named after him and is known as the Sobolev system.

Here,

- \( \bar{v} = (v_1, v_2, v_3) \) - velocity of an incompressible and nonviscous fluid,
- \( \bar{\omega} = (0, 0, \omega) \) - constant angular velocity,
- \( P \) - pressure, and
- \( \bar{f} = (f_1, f_2, f_3) \) - external force.

The Cauchy problem (1), (2) was considered in the domain \( \Omega = \{(x, t) \mid x \in \mathbb{R}^3, t \geq 0\} \).

Various initial-boundary value problems for the Sobolev system and other similar systems with viscosity or/and compressibility had been studied by other mathematicians (Maslennikova 1968, Maslennikova and Bogovskii 1977, Maslennikova and Bogovskii 1975).

Here, in this work, the author has undertaken for study a system of partial differential equations similar to the Sobolev system, where the heat conduction is also taken into account. Such a system, generally, occurs in the dynamics of atmosphere and oceans. The system is as follows:
where,
\[ v_1, v_2, v_3 \] - components of velocity vector as in the Sobolev system.
\[ p \] - pressure,
\[ T \] - temperature (declination of temperature from some value \( T_0 \) corresponding to the plane \( x_3 = 0 \)),
\[ \sigma \] - coefficient of heat conduction (positive constant),
\[ \gamma \] - mean gradient of density (positive constant),
\[ \omega \] - constant vector of angular velocity,
\[ f_1, f_2, f_3 \] - components of external force,
\[ f \] - continuous function of \( x \) and \( t \).

The initial conditions are:
\[
\begin{align*}
\mathbf{v}(x, t)|_{t=0} &= \mathbf{v}_0(x) \\
T(x, t)|_{t=0} &= T_0(x)
\end{align*}
\] (3)

where,
\[ \mathbf{v} = (v_1, v_2, v_3), \text{ and } \]
\[ \mathbf{v}_0 = \left( v^0_1, v^0_2, v^0_3 \right). \]

Solution of the Cauchy problem

The solution of the corresponding homogeneous system with initial conditions (4) in the domain
\[ \Omega = \{(x, t) \mid x \in \mathbb{R}^3, t \geq 0\} \], has been found in explicit form by the author in (Adhikary 1997).

The solution is given by
\[
\begin{align*}
v_1(x, t) &= \int_{\mathbb{R}^3} \Delta v_1^0(y) \kappa_1(x-y, t) \, dy + \int_{\mathbb{R}^3} \left( \sigma \gamma \frac{\partial^2 v_1^0(y)}{\partial y_1 \partial y_2} - \sigma \gamma \frac{\partial^2 v_2^0(y)}{\partial y_1 \partial y_2} + \sigma \omega \frac{\partial^2 T_0(y)}{\partial y_1 \partial y_2} \right) \kappa_2(x-y, t) \, dy \\
&\quad + \int_{\mathbb{R}^3} \left( \sigma \gamma \frac{\partial^2 v_1^0(y)}{\partial y_2 \partial y_3} - \sigma \gamma \frac{\partial^2 v_2^0(y)}{\partial y_1 \partial y_2} + \sigma \omega \frac{\partial^2 T_0(y)}{\partial y_1 \partial y_3} \right) \kappa_3(x-y, t) \, dy
\end{align*}
\]

where,
\[ \kappa_1(x, t) = \frac{1}{2 \pi^2 r} \int_0^{\pi/2} \cos \left( t \, g \left( \psi \right) \right) \, d\psi \]
\[ \kappa_2(x, t) = \frac{1}{2 \pi^2 r} \int_0^{\pi/2} \sin \left( t \, g \left( \psi \right) \right) \, d\psi \]
with
\[ g(\psi) = \sqrt{(\omega^2 - \sigma \gamma) \left( \frac{\rho}{\omega^2} \right)^2 \sin^2 \psi + \sigma \gamma} \]
\[ \rho = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \]
and
\[ r = \sqrt{r^2 + (x_3 - y_3)^2} \].

\( v_2(x, t), v_3(x, t), P(x, t) \) and \( T(x, t) \) have similar expressions as that of \( v_1(x, t) \).

The solution of the nonhomogeneous system with the same initial conditions in the same domain can be found from the solution of the corresponding homogeneous system by using Duhamel's principle (Courant 1962). In this connection, for the external force
\[ \vec{F} = (f_1, f_2, f_3) \in L^2(\mathbb{R}^3), \]
we assume, without loss of generality (Maslennikova 1977), that
\[ \text{div} \vec{F} = 0. \]

The solution of the Cauchy problem for the nonhomogeneous system with conditions (4) and (5) has the following form.
\[ v_1^*(x, t) = v_1(x, t) + \bar{v}_1(x, t), \]
where,
\[ \bar{v}_1(x, t) = \int_0^t \int_{\mathbb{R}^3} \Delta f_1(y, \tau) \kappa_1(x - y, t - \tau) \, dy \, d\tau \]
\[ + \int_0^t \int_{\mathbb{R}^3} \left\{ \omega \frac{\partial^2 f_2(y, \tau)}{\partial y_3^2} - \omega \frac{\partial^2 f_2(y, \tau)}{\partial y_2 \partial y_3} + \sigma \frac{\partial^2 f_2(y, \tau)}{\partial y_1 \partial y_3} \right\} \kappa_2(x - y, t - \tau) \, dy \, d\tau \]
\[ + \int_0^t \int_{\mathbb{R}^3} \left\{ \sigma \frac{\partial^2 f_1(y, \tau)}{\partial y_2^2} - \sigma \frac{\partial^2 f_1(y, \tau)}{\partial y_1 \partial y_2} + \omega \frac{\partial^2 f_1(y, \tau)}{\partial y_2 \partial y_3} \right\} \kappa_3(x - y, t - \tau) \, dy \, d\tau. \]

Expressions for \( v_2^*(x, t), v_3^*(x, t), P^*(x, t) \) and \( T^*(x, t) \) are found exactly in the same way as that in \( v_1^*(x, t) \).

**Uniqueness of the solution**

The following uniqueness theorem for the solutions of the Cauchy problem (3), (4) is valid.

**Theorem**

The solutions \( \bar{v}(x, t) \) and \( T(x, t) \) of the Cauchy problem (3), (4) with \( \vec{F} = 0 \) are unique in \( L^2 \), while the solution \( P(x, t) \) is determined up to a function of \( t \). In addition, \( \nabla P \) is again unique in \( L^2 \).
Proof

The theorem can be proved using the energetic estimation that we obtain below assuming the validity of the operations performed.

Let us multiply the first three equations of system (3) with \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) respectively and the fourth equation by \( \gamma T \) to obtain the following equations:

\[
\begin{align*}
\frac{1}{2} \frac{\partial \vec{v}_1^2}{\partial t} - \omega \vec{v}_1 \vec{v}_2 + \frac{\partial P}{\partial x_1} \vec{v}_1 &= 0 \\
\frac{1}{2} \frac{\partial \vec{v}_2^2}{\partial t} + \omega \vec{v}_1 \vec{v}_2 + \frac{\partial P}{\partial x_2} \vec{v}_2 &= 0 \\
\frac{1}{2} \frac{\partial \vec{v}_3^2}{\partial t} + \frac{\partial P}{\partial x_3} \vec{v}_3 + \sigma T \vec{v}_3 &= 0 \\
\frac{1}{2} \frac{\partial T^2}{\partial t} - \sigma T \vec{v}_3 &= 0
\end{align*}
\]

On addition, we get

\[
\frac{1}{2} \left[ \frac{\partial \vec{v}_1^2}{\partial t} + \frac{\partial \vec{v}_2^2}{\partial t} + \frac{\partial \vec{v}_3^2}{\partial t} \right] + \frac{1}{2} \frac{\sigma}{\gamma} \frac{\partial T^2}{\partial t} + \frac{\partial P}{\partial x_1} \vec{v}_1 + \frac{\partial P}{\partial x_2} \vec{v}_2 + \frac{\partial P}{\partial x_3} \vec{v}_3 = 0.
\]

Therefore, on integrating over the region \( Q_\tau = \{(x, t): x \in \mathbb{R}^3, 0 < t < \tau\} \),

\[
\frac{1}{2} \int_{Q_\tau} \frac{\partial \vec{v}_1^2}{\partial t} \ dx \ dt + \frac{1}{2} \frac{\sigma}{\gamma} \int_{Q_\tau} \frac{\partial T^2}{\partial t} \ dx \ dt + \int_{Q_\tau} (\nabla P \cdot \vec{v}) \ dx \ dt = \text{const.}
\]

and hence,

\[
\int_{\Omega} \left( \vec{v}_1 \vec{v}_2 + \frac{\sigma}{\gamma} T^2 \right) \ dx \ dt = \int_{\Omega_0} \left( \vec{v}_0 \vec{v}_2 + \frac{\sigma}{\gamma} T_0^2 \right) \ dx \tag{6}
\]

since the last term of left hand side is zero.

Assume that there exist two distinct solutions

\{ \vec{v}^* (x, t), T^* (x, t) \} and \{ \vec{v}^{**} (x, t), T^{**} (x, t) \} satisfying the system (3) and the initial conditions (4) in usual generalized sense.

Let \( \vec{v} = \vec{v}^* - \vec{v}^{**} \) and \( T = T^* - T^{**} \) be their differences. In order to prove that the above problem has at most one solution, we have to show that \( \vec{v} = 0 \) and \( T = 0 \).

We see that

\[
\begin{align*}
\vec{v}_{t=0} &= \vec{v}_0 = 0 \\
T_{t=0} &= T_0 = 0
\end{align*}
\]

i.e. \( \vec{v} \) and \( T \) satisfy the zero initial values.
Therefore, from (6), we get:
\[ \int_{\Omega} \left( \tilde{\nu}^2 + \frac{\sigma}{\gamma} T^2 \right) \, dx \, dt = 0 \quad (8) \]

Since \( \sigma, \gamma > 0 \), the integrand in (8) is non-negative. Moreover, it is continuous over the region \( \Omega \). Hence it must be identically zero. That means \( \tilde{\nu} = 0 \) and \( T = 0 \). Consequently, \( \tilde{\nu}^* = \tilde{\nu}^{**} \) and \( T^* = T^{**} \). Hence, the uniqueness of the solution is established.

Since the initial Cauchy data for \( P(x, t) \) are not given, therefore, \( P(x, t) \) is determined within a term depending on \( t \). Uniqueness of \( \nabla P \) can be shown in the same way as that of \( \tilde{\nu} \) and \( T \).

**Estimation of the solution in \( L^p \)**

To obtain the estimation of the solution in \( L^p \), we use Marcinkiewicz theorem on multiplicators (Lizorkin 1969, Maslennikova and Bogovski 1975). First we need some definitions.

**Definition 1**

The space \( L^p(\Omega) \) is defined as the collection of all functions \( f \) specified on \( \Omega \) for which the norm
\[ \| f \|_{L^p(\Omega)} = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty \]

is finite.

**Definition 2**

The Sobolev space \( W^m_p(\Omega) \) is defined as
\[ W^m_p(\Omega) = \{ f \in L^p(\Omega) \colon D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq m \} \]

In other words, \( W^m_p(\Omega) \) is the collection of all functions in \( L^p(\Omega) \) such that all distributional derivatives up to order \( m \) are also in \( L^p(\Omega) \).

We define the norm in \( W^m_p(\Omega) \) as follows:
\[ \| f \|_{W^m_p(\Omega)} = \left\{ \sum_{|\alpha| \leq m} \| D^\alpha f \|_{L^p(\Omega)}^p \right\}^{1/p} \]

where, \( \| \cdot \|_{L^p(\Omega)} \) denotes the \( L^p \) norm.

We denote by \( W^{k,l}_{p,t,x} \) the Sobolev space of functions having \( k \) derivatives with respect to \( t \) and \( l \) derivatives with respect to \( x \), which are \( p \)-th power summable.

**Definition 3**

The space of all Fourier transformations of generalized functions \( T \in S' \) that determine bounded translations of \( L^p \) onto itself is denoted by \( M^p_p \). The elements of \( M^p_p \), say \( \hat{T} \) are called *multiplicators* of type \((p, p)\). The set \( M^p_p \) becomes a Banach space, if the norm of \( \hat{T} \) is taken to be the infimum of the constants \( C \) for which
\[ \| T * \varphi \|_p \leq C \| \varphi \|, \quad \varphi \in S. \]

Now, we state the Marcinkiewicz's theorem on multiplicators (Lizorkin 1969).
Theorem 1

Let a function $\Phi(\xi)$ and its partial derivatives
\[ \frac{\partial^m \Phi}{\partial \xi_1 \cdots \partial \xi_m} \quad (m \geq 1; k_1, k_2, \ldots, k_m - \text{distinct}) \]
be continuous and satisfy the condition
\[ \left| \frac{\partial^m \Phi}{\partial \xi_1 \cdots \partial \xi_m} \right| \leq M \]
Then, $\Phi$ becomes a multiplicator of type $(p, p)$ with the norm whose upper bound is $M$.

i.e. $\|\Phi\|_p \leq C M \|\phi\|_p, \phi \in S$.

For detailed proof of this theorem, see Lizorkin (1969).

Onwards, we assume that $1 < p < \infty$. The following theorem holds true.

Theorem 2

If the initially given functions $\varphi^0(x), T^0(x) \in W^k_p(\mathbb{R}^3)$, then the following a priori estimates for the solution of the Cauchy problem (3), (4) will take place:
\[ \|\hat{v}\|_{W^k_{p,t,x}} + \|T\|_{W^k_{p,t,x}} \leq C \left( \|\varphi^0\|_{W^k_{p}} + \|T^0\|_{W^k_{p}} \right) \]
where, $\mathbb{R}^4_H = \{(x, t) : x \in \mathbb{R}^3, 0 \leq t \leq H\}$.

Proof

Let us consider the Fourier representation of the solution of the problem (3), (4) with $\hat{F} = 0$ obtained in (Adhikary 1997). According to Marcinkiewicz theorem stated above, the functions
\[ \cos \left( \frac{\xi_1 \phi(\xi)}{\xi_1} \right), \sin \left( \frac{\xi_1 \phi(\xi)}{\xi_1} \right) \]
and their products with $\frac{\xi_1 \phi(\xi)}{\xi_1} \phi(\xi)$ become the multiplicators of type $(p, p)$. Consequently, theorem 2 holds.

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